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Instability of generalised AdS black holes and thermal field theory

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Abstract: We study black holes in AdS-like spacetimes, with the horizon given by an arbitrary positive curvature Einstein metric. A criterion for classical instability of such black holes is found in the large and small black hole limits. Examples of large unstable black holes have a Böhm metric as the horizon. These, classically unstable, large black holes are locally thermodynamically stable. The gravitational instability has a dual description, for example by using the $AdS_7 \times S^4$ version of the AdS/CFT correspondence. The instability corresponds to a critical temperature of the dual thermal field theory defined on a curved background.

Keywords: AdS-CFT and dS-CFT Correspondence, Black Holes.

JHEP08(2003)019

Contents

1. Introduction

Gravitational physics in higher dimensions, and black hole physics in particular, allows a wealth of phenomena that are not possible in the more tightly constrained dynamics of four dimensional spacetime. This paper will study one such phenomenon in the context of the correspondence between gravity in Anti-de Sitter (AdS) spacetime and gauge theories in one less dimension [1]–[3]. Black holes play an important rôle in understanding this correspondence at finite temperature [4]. Anti-de Sitter black hole spacetimes contain information about the thermodynamics of a dual field theory. For standard AdS black holes, the dual field theory is defined on the background $S^1 \times S^d$. For the generalised black holes to be introduced shortly, the field theory is instead on the background $S^1 \times B'$, with B' an arbitrary Einstein manifold.

We will exhibit an instability of generalised black holes with a negative cosmological constant and predict a corresponding effect in the dual thermal field theory. The most immediate embedding of these results into a known duality turns out to be into the correspondence relating M-theory on $AdS_7 \times S^4$ and the six dimensional conformal field theory describing M5 branes at low energy [1]. The paper has three parts:

- Derivation of the criterion for classical instability of generalised black holes with a negative cosmological constant (sections 2–4).
- Discussion of the relationship between classical instability and thermodynamic stability of these black holes (section 5).
- Discussion of the dual description of the instability in thermal field theory (section 6).

Let us briefly motivate these three points in turn.

One important feature of higher dimensional gravity is that there are many possibilities for constructing non-asymptotically flat spacetimes. Asymptotic flatness itself in higher dimensions appears to be more subtle than in four dimensions [5]. The natural D-dimensional generalisation of the four dimensional Schwarzschild black hole, with vanishing cosmological constant, was written down many years ago [6]. One may then prove that this is the unique regular static black hole that is asymptotically flat [7, 8]. However, if one drops the requirement of asymptotic flatness, then the Einstein equations allow the replacement of the sphere that forms the horizon with any positive curvature Einstein manifold, B, appropriately normalised. This B is related to the B' on which the dual field theory is defined by a rescaling of the metric. Even if one wishes to retain the spherical topology of the horizon, a countable infinity of such metrics are known on spheres of dimension $d = 5 \ldots 9$, constructed by Böhm [9]. With a negative cosmological constant, other possibilities exist, in which the horizon metric may have negative or zero curvature [10]–[13], but these will not be considered here.

A natural question concerns the stability of these new black hole spacetimes. Classical instabilities of generalised black holes in flat space, i.e. with a vanishing cosmological constant, were studied in some detail in [14]. A criterion for instability was found that depended on the lowest Lichnerowicz eigenvalue of the horizon manifold, B. If the eigenvalue is less than a critical value, the spacetime is unstable.

The unstable mode in question is a transverse tracefree tensor harmonic of the horizon metric, other modes are not expected to be dangerous [14]. Such modes do not exist in four spacetime dimensions, because the horizon S^2 does not admit nonzero tensor harmonics [15], so the instability is inherently higher dimensional. In seven spacetime dimensions, examples of stable flat space black holes are given when the horizon manifold is an Einstein-Sasaki manifold, such as $T^{1,1}$ or S^5 [16]. Examples of unstable black holes in this dimension are provided when the horizon is a Böhm metric on S^5 or $S^2 \times S^3$ [16]. The possibility of both stable and unstable generalised black holes shows that a naïve global thermodynamic argument [8], that any positive curvature horizon must have volume and hence entropy lower than a spherical horizon [17], is insufficient to understand classical instability. For example, some of the stable Einstein-Sasaki metrics on $S^2 \times S^3$ have much lower volume than the unstable Böhm metrics on the same topology [18, 16].

This paper considers the classical stability of generalised black holes in Anti-de Sitter space, that is, with a negative cosmological constant. In section 2 the formalism is set up and the perturbation mode is described. In section 3 an instability criterion is derived in the large black hole limit. The dependence on the horizon size is found analytically whilst the dependence on the dimension is found numerically. Larger black holes are more likely to be stable. In particular, for a fixed horizon manifold, there is always a critical size above which black holes are stable. On the other hand, for a fixed, arbitrarily large horizon size, there exists a Böhm metric that one may use as the horizon manifold, such that the resulting black hole is unstable. In section 4 small black holes are discussed and it is shown that the criterion for instability of small generalised black holes is the same as the criterion in flat space, as one might expect.

Recent work on large charged black holes in Anti-de Sitter space lead to the conjecture by Gubser and Mitra that translationally invariant black branes are classically stable if and only if they are locally thermodynamically stable [19, 20]. This conjecture has since been strengthened and generalised [21, 22]. Local thermodynamic stability is the statement that the hessian of the entropy with respect to extensive quantities is negative definite. In the uncharged case that we are interested in, this reduces to the requirement of positive specific heat. We see in section 5 that the unstable large black holes are locally thermodynamically stable. This result is discussed in the context of the Gubser-Mitra conjecture. We conclude that, unlike spherical black holes, the large generalised black hole limit cannot be thought of as a translationally invariant black brane limit, and therefore the conjecture is not contradicted. However, the situation suggests that one has to be cautious about using thermodynamic arguments to explain classical instabilities.

Finally, in section 6 we discuss the dual field theory implications of the instability. The geometry of the generalised black holes at a fixed radius is $S^1 \times B'$, where B' is just B rescaled by a factor of the radius squared. In particular, this is true at large radius and is therefore the background on which the dual thermal field theory is defined. The S^1 is of course the time direction. We will see that the instability is naturally translated into the existence of a critical temperature in the field theory. For various reasons, it is difficult to establish exactly what occurs at the critical temperature in the field theory. This situation will suggest various directions for future research.

2. Generalised AdS black holes and a tensor perturbation

A D dimensional black hole spacetime has metric

$$ds^2_D = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\tilde{s}^2_d \,, \tag{2.1}$$

where $d\tilde{s}^2_d$ is a riemannian metric on a $d = D-2$ dimensional manifold B, call it the horizon manifold. The black holes will be called generalised because B need not be a round sphere. We are interested in the stability of such generalised black hole spacetimes.

To study the stability of a solution to Einstein's equations, consider a small transverse tracefree perturbation to the metric. The first order change in the Ricci tensor is

$$\begin{aligned} g_{ab} &\to g_{ab} + h_{ab}, \quad \text{such that} \quad h^a{}_a = \nabla^a h_{ab} = 0 \\ R_{ab} &\to R_{ab} + \frac{1}{2}(\Delta_L h)_{ab} \,, \end{aligned} \tag{2.2}$$

where the Lichnerowicz operator acting on a symmetric second rank tensor h is

$$(\Delta_L h)_{ab} = 2R^c{}_{abd}h^d{}_c + R_{ca}h^c{}_b + R_{cb}h^c{}_a - \nabla^c\nabla_c h_{ab} \,. \tag{2.3}$$

Gauge freedom was discussed in [14] and shall not be of concern here.

Linearised perturbations to black hole spacetimes may be separated into scalar, vector and tensor modes with respect to the horizon. It was shown in [14] that the dangerous mode for generalised black holes spacetimes is a tensor mode, which satisfies the further

JHEP08(2003)019

conditions

$$h_{0a} = h_{1a} = 0\,, \tag{2.4}$$

where $0, 1$ are the t, r coordinates. The conditions (2.4) and the form of the metric (2.1) imply that the transverse tracefree property of h_{ab} (2.2) is inherited by $h_{\alpha\beta}$. The indices $a, b, \ldots$ run from $0 \ldots D$ and the indices $\alpha, \beta, \ldots$ will run from $2 \ldots D$, the coordinates on B. Thus the mode is a tensor mode on the d-dimensional horizon manifold B.

Unstable modes are bounded, normalisable modes [14] that grow in time,

$$h_{\alpha\beta} = \widetilde{h}_{\alpha\beta}(x) r^2 \varphi(r) e^{\omega t}\,, \tag{2.5}$$

where x are coordinates on B and we have decomposed the mode into Lichnerowicz harmonics on the horizon manifold, B,

$$(\widetilde{\Delta}_L \tilde{h})_{\alpha\beta} = \lambda \widetilde{h}_{\alpha\beta}\,. \tag{2.6}$$

Tildes denote tensors on B. Thus $\widetilde{\Delta}_L$ is the Lichnerowicz operator on B. Note that although the mode (2.5) only has legs along the horizon directions, it has a radial dependence, transverse to the horizon. Radial boundary conditions, at the horizon and at infinity, are discussed below.

We are presently interested in generalised black holes in Anti-de Sitter spacetime. Thus the metric (2.1) must solve the vacuum Einstein equations with a negative cosmological constant

$$R_{ab} = -H^2(d+1) g_{ab}\,. \tag{2.7}$$

We will be considering the case when the metric on B is Einstein with positive curvature

$$\tilde{R}_{\alpha\beta} = (d-1)\tilde{g}_{\alpha\beta}\,. \tag{2.8}$$

Again, tildes denote tensors on B. The normalisation corresponds to having the same scalar curvature as S^d. To satisfy the Einstein equations, incorporating (2.8), the function f must be of the form

$$f(r) = 1 - \left(\frac{\alpha}{r}\right)^{d-1} + H^2 r^2\,. \tag{2.9}$$

Here α^{d-1} is proportional to the mass of the black hole.

The perturbation must satisfy the linearised equation $\delta R_{\alpha\beta} = -H^2(d+1)h_{\alpha\beta}$. This gives an equation for φ that may be written as a Schrödinger equation [14] by changing variables to Regge-Wheeler type coordinates and rescaling

$$dr_* = \frac{dr}{f}\,, \qquad \Phi = r^{d/2}\varphi\,. \tag{2.10}$$

The equation for the perturbation becomes

$$-\frac{d^2\Phi}{dr_*^2} + V(r(r_*))\Phi = -\omega^2\Phi \equiv E\Phi\,, \tag{2.11}$$

JHEP08(2003)019

where the potential is

$$V(r)=\frac{\lambda f}{r^2}+\frac{d-4}{2}\frac{f'f}{r}+\frac{d^2-10d+8}{4}\frac{f^2}{r^2}+2H^2(d+1)f\,. \tag{2.12}$$

The stability problem reduces to the existence of bound states with $E<0$ of the Schrödinger equation with potential $V(r(r_*))$. If such a bound state of the Schrödinger equation exists, then the spacetime (2.1) is unstable to modes of the form (2.5). That is to say, there will be an instability if the ground state eigenvalue, E_0, of (2.11) is negative.

The mode must also satisfy boundary conditions at the horizon and at radial infinity. Firstly, the mode must remain bounded, so that the linearised approximation is valid. Secondly, one requires that the mode is normalisable, $\int\Phi^2 dr_*<\infty$. This second issue was shown in [14] to be equivalent to requiring finite energy of the spacetime mode (2.5). In practice, for undesirable modes $\Phi(r)$ diverges at the horizon or at infinity. These issues are discussed in some detail in [14]. All modes considered in this work are well-behaved.

Explicitly, for the case of an AdS black hole, the potential becomes

$$\begin{aligned}V(r)\;=\;&\left[\frac{d^2-10d+8+4\lambda}{4r^2}+\frac{(d^2+2d)H^2}{4}\right](H^2r^2+1)+\\&+\frac{1}{\alpha^2}\left(\frac{\alpha}{r}\right)^{d+1}\left[\frac{10d-8-4\lambda-2dH^2r^2}{4}-\left(\frac{\alpha}{r}\right)^{d-1}\frac{d^2}{4}\right].\end{aligned} \tag{2.13}$$

It is clear that an analytic solution to the Schrödinger equation is out of the question. Further, the ideas used in [14], which may be made rigorous, to derive a criterion for instability for generalised flat space black holes are not so easily applied in this case. Another argument is needed.

3. A criterion for instability

Let us introduce the horizon radius r_+, satisfying $f(r_+)=0$. In terms of this radius, the function (2.9) becomes

$$f(r)=1-\frac{(1+H^2r_+^2)r_+^{d-1}}{r^{d-1}}+H^2r^2\,. \tag{3.1}$$

To be in a regime in which gravity is expected to be a valid low energy theory, we need the spacetime curvature to be small outside the horizon, which requires H to be small. In this section we will further restrict attention to large black holes with

$$Hr_+\gg 1\,. \tag{3.2}$$

Although we are considering a classical gravitational instability here, we will later be interested in finding a dual field theory description. In that context, we want to be in the regime in which the euclidean black hole makes the dominant contribution to the quantum gravity partition function [23, 4], as opposed to periodically identified, generalised euclidean AdS space. The large black hole condition (3.2) ensures that we are safely within this regime.

JHEP08(2003)019

d	1	2	3	4	5	6
$-\frac{\lambda_c}{H^2 r_+^2}$	4	7.4080	11.588	16.494	22.097	28.375
d	7	8	9	10	11	12
$-\frac{\lambda_c}{H^2 r_+^2}$	35.313	42.897	51.118	59.966	69.434	79.515

Table 1: Critical eigenvalues for various dimensions.

More will be said about the large black hole limit below. Small black holes, of less interest to us here, will be discussed in the following section.

The criterion for classical dynamical instability will be a critical value for the lowest eigenvalue of the Lichnerowicz operator on the horizon manifold B [14]. If the eigenvalue is below this critical value, then the potential (2.13) admits a bound state with negative energy. As we raise the eigenvalue towards the critical value, the energy of the bound state is raised to zero. Thus to find the critical value itself λ_c, set $\omega = 0$. Note that this wouldn't have worked for flat space black holes, because in that case the potential remains negative at large radial direction, and there is no zero energy bound state.

Using (3.2) in the Schrödinger equation (2.11) and further putting $\omega = 0$, one obtains another Schrödinger type equation, now with a weight function,

$$-\frac{d^2\Phi}{d\rho_*^2} + \widetilde{V}(\rho(\rho_*))\Phi = \frac{-\lambda_c}{H^2 r_+^2} k(\rho(\rho_*))\Phi\,, \tag{3.3}$$

where we have set $\rho = r/r_+$, so that the horizon is now at $\rho = 1$, and

$$\begin{aligned} \frac{d\rho}{d\rho_*} &= \rho^2 - \frac{1}{\rho^{d-1}}\,, \\ \widetilde{V}(\rho) &= \frac{1}{4}\left[\rho^2 - \frac{1}{\rho^{d-1}}\right]\left[2d + d^2 + \frac{d^2}{\rho^{d+1}}\right], \\ k(\rho) &= 1 - \frac{1}{\rho^{d+1}}\,. \end{aligned} \tag{3.4}$$

We are now looking for the lowest bound state of this new Schrödinger equation (3.3). This will give us λ_c and hence the criterion for instability. Note that the potential is positive and further $Hr_+ \gg 1$; therefore λ_c will be large and negative.

It seems that this equation cannot be solved analytically except for the case when $d = 1$. However, there are now no undetermined parameters in the potential and so it is not difficult to find the lowest eigenvalue numerically. The results for a range of dimensions are shown in table 1.

One may do a least squares fit to this data using a quadratic function in d. To get reliable statistics, we consider the numerically found eigenvalues for 110 values for d, spaced evenly between $d = 1$ and $d = 12$. The result is

$$\lambda_c \approx -H^2 r_+^2 \left[0.86 + 2.6d + 0.33d^2\right]. \tag{3.5}$$

The fit is good in the range considered, see figure 1 below, although less good for the smaller values of d. A quadratic fit is natural because higher powers of d would have very small coefficients.

JHEP08(2003)019

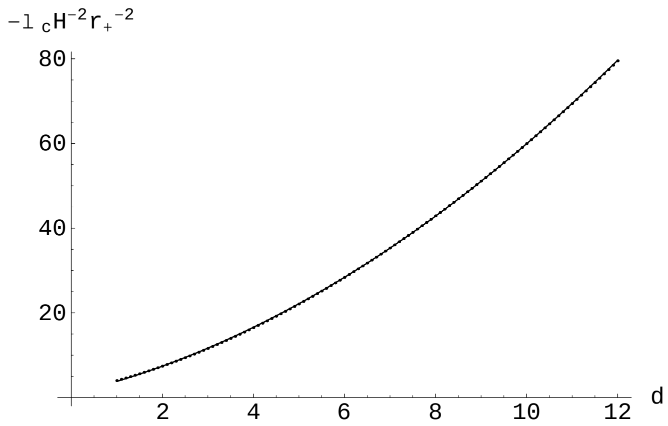

Figure 1: $-\frac{\lambda_c}{H^2 r_+^2}$ against d, datapoints and fitted curve shown.

The criterion for instability is thus, for $Hr_+ \gg 1$, whether the horizon manifold B admits a Lichnerowicz eigenvalue lower than the critical value (3.5),

$$\lambda < \lambda_c \quad \Leftrightarrow \quad \text{Instability}\,. \tag{3.6}$$

The existence and properties of Böhm metrics [9] show that this instability criterion is not vacuous. Some properties of these metrics are reviewed in the second half of section 5 of this paper. The essentials are as follows. The Böhm metrics are infinite sequences of nonsingular inhomogeneous Einstein metrics on $S^p \times S^{d-p}$ and S^d for $5 \le d \le 9$ and $p \ge 2$, $d-p \ge 2$. These metrics may therefore be used on horizon manifolds. It was shown in [16] that the sequence of Böhm metrics on $S^2 \times S^3$ and S^5 admit an increasingly negative Lichnerowicz eigenvalue. For example, the metric on S^5 denoted Böhm$(2,2)_6$ in [16] has a Lichnerowicz eigenvalue lower than -1040, assuming the curvature normalisation of (2.8). Other metrics on S^5 in the sequence, denoted Böhm$(2,2)_{2m}$, are expected to have arbitrarily lower Lichnerowicz eigenvalues. Thus, whatever value of Hr_+ is needed in order for the approximation (3.2) to be valid, there will be a Böhm metric with a Lichnerowicz eigenvalue lower than the corresponding critical value of equation (3.5). The generalised black hole spacetime with this horizon manifold will then be unstable.

As we increase m in the sequence of metrics Böhm$(2,2)_{2m}$, the metrics, whilst remaining nonsingular, have increasingly large curvature at two points. They tend to the singular "double cone" in the limit $m \to \infty$. However, we will see in section 5 that by considering a double scaling limit, in which both the black hole size and the curvature on the horizon manifold are scaled, one may always obtain unstable Böhm black holes with a maximum spacetime curvature that is well within the regime of validity for classical gravity.

4. Small black holes

Normalisability of the perturbation in the infinite spacetime volume implies that the instability is localised near the event horizon of the black hole. Therefore, we should expect that for small generalised black holes, $Hr_+ \ll 1$, the criterion for classical instability will just be the flat space criterion found in [14]. In this section we check that the flat space instability criterion is indeed recovered. Independently of the classical instability criterion, small black holes are thermodynamically unstable and do not contribute dominantly to the quantum gravity partition function [23, 4].

The strategy to find the critical Lichnerowicz eigenvalue will be to separately solve the Schrödinger equation (2.11) with $\omega = 0$ in the regions $\frac{r}{r_+} \gg 1$ and $\frac{r}{r_+} \ll \frac{1}{Hr_+}$. Because Hr_+ is small, both solutions will be valid in the non-empty overlap $1 \ll \frac{r}{r_+} \ll \frac{1}{Hr_+}$. A bound state will exist if the well-behaved solution at infinity matches onto a well-behaved solution at the horizon.

In the region $\frac{r}{r_+} \ll \frac{1}{Hr_+}$, or equivalently $Hr \ll 1$, the Schrödinger equation (2.11) reduces to the flat space equation with $H = 0$. This is immediately seen from equations (2.9) and (2.13). The solution to this equation that is regular at the horizon may be found in terms of a hypergeometric function

$$\Phi_1(r) = \left(\frac{r}{r_+}\right)^{d/2} {}_2F_1\left(\frac{1}{2} + \frac{Q}{2(d-1)}, \frac{1}{2} - \frac{Q}{2(d-1)}, 1; 1 - \left(\frac{r}{r_+}\right)^{d-1}\right), \tag{4.1}$$

where $Q = \sqrt{4\lambda - 16 + (d-5)^2}$.

Now consider the region $\frac{r}{r_+} \gg 1$. In this region, the Schrödinger equation (2.11) becomes the equation in pure generalised AdS, with no black hole present, e.g. with $r_+ = 0$. This is easily seen from equation (3.1). The solution to this equation that decays at infinity was written down in [14] and is again given in terms of a hypergeometric function

$$\Phi_2(r) = r^{-(2+d)/2} {}_2F_1\left(\frac{3+d+Q}{4}, \frac{3+d-Q}{4}, \frac{6+2d}{4}; \frac{-1}{H^2r^2}\right), \tag{4.2}$$

where as before $Q = \sqrt{4\lambda - 16 + (d-5)^2}$.

To match these two solutions in the overlapping region of validity, require

$$\lim_{\frac{r}{r_+} \gg 1} \Phi_1(r) \propto \lim_{Hr \ll 1} \Phi_2(r) \,. \tag{4.3}$$

These limits may be calculated using standard properties of hypergeometric functions. We find

$$\begin{aligned}
Q \neq 0 \Rightarrow & \begin{cases} \Phi_1(r) \sim r^{(1+Q)/2} & \text{for} \quad \dfrac{r}{r_+} \gg 1, \\ \Phi_2(r) \sim r^{(1-Q)/2} & \text{for} \quad Hr \ll 1, \end{cases} \\
Q = 0 \Rightarrow & \begin{cases} \Phi_1(r) \sim r^{1/2} \ln r & \text{for} \quad \dfrac{r}{r_+} \gg 1, \\ \Phi_2(r) \sim r^{1/2} \ln r & \text{for} \quad Hr \ll 1, \end{cases}
\end{aligned} \tag{4.4}$$

Thus, the two limits generically agree if and only if $Q = 0$. Note that Q pure imaginary will generally not match because the two solutions will not be in phase. The instability criterion is therefore

$$\lambda < \lambda_c = 4 - \frac{(d-5)^2}{4} \quad \Leftrightarrow \quad \text{Instability}\,. \tag{4.5}$$

This is the same criterion as was found for flat space black holes [14].

JHEP08(2003)019

Interpolating between our two results on critical Lichnerowicz eigenvalues gives a picture for the behaviour of λ_c as a function of the black hole size, Hr_+. For small black holes, the critical eigenvalue is a constant (4.5). As the black hole gets larger, the critical eigenvalue is lowered. For large black holes the critical eigenvalue is increasingly negative (3.5).

A useful way of thinking about the instability is as follows. Suppose a black hole with a given horizon manifold is unstable in the small black hole limit. As we vary the black hole size, there will be a critical value such that if the black hole is smaller than the critical size it will be unstable and if it is larger it will be stable. Note that the Lichnerowicz spectrum of an Einstein metric on a compact manifold admits a lower bound in terms of the Weyl curvature tensor [24, 16], so such a critical size, above which the black hole is stable, always exists.

5. Thermodynamics and the black brane limit

In this section it will be seen that our classically unstable large black holes are locally thermodynamically stable. For standard AdS black holes, where the horizon is a round sphere, the physics in the large size limit is locally that of a black brane, in which the horizon may be considered to be noncompact. A recent conjecture of Gubser and Mitra states that translationally invariant black branes are locally thermodynamically stable if and only if they are classically stable [19]–[21]. To see whether we have just found counterexamples to this conjecture, we need to understand whether the large generalised black holes may be locally understood as translationally invariant black branes.

The thermodynamics of the generalised AdS black holes we are considering is the same as for standard AdS black holes. The large black hole limit, $Hr_+ \gg 1$, together with the small curvature condition $H \ll 1$, implies that $r_+ \gg 1$. In this limit the thermodynamic quantities associated with AdS black holes, temperature (T), energy (M), entropy (S) and heat capacity (C), are well known, e.g. [4],

$$\begin{aligned} T &\approx \frac{(d+1)H^2 r_+}{4\pi}\,, \\ M = E &\approx \frac{d\mathrm{Vol}(B)H^2 r_+^{d+1}}{16\pi G}\,, \\ S &\approx \frac{\mathrm{Vol}(B) r_+^d}{4G}\,, \\ C = \frac{dE}{dT} &\approx \frac{d\mathrm{Vol}(B) r_+^d}{4G} \approx dS\,. \end{aligned} \tag{5.1}$$

The last of these quantities shows that we are comfortably in the regime where the heat capacity is positive and hence the black hole is locally thermodynamically stable.

For the standard AdS black holes with spherical horizons, the large black hole limit is locally the black brane limit. Qualitatively, this is because all curvatures at the horizon are negligible in the large black hole limit and the sphere, S^d, may be locally considered to be noncompact flat space, $\mathbb{R}^d$.

The situation for generalised AdS black holes is a little more subtle. In the limit where $Hr_+ \gg 1$ the black hole metric may be written as

$$
\begin{aligned}
ds^2 &= \rho^2 \left[-g(\rho) H^2 r_+^2 dt^2 + r_+^2 d\tilde{s}_d^2\right] + \frac{d\rho^2}{\rho^2 H^2 g(\rho)}\,, \\
g(\rho) &= 1 - \rho^{-d-1}\,,
\end{aligned}
\tag{5.2}
$$

where $\rho = r/r_+$, so the horizon is at $\rho = 1$. If $B = S^d$ one can now locally replace $r_+^2 d\tilde{s}_d^2$ by the flat metric $dx_1^2 + \cdots + dx_d^2$, because H is small so r_+ is large. The resulting spacetime (5.2) then describes a black brane with d noncompact, translationally invariant, spatial directions and one transverse direction. We will call $r_+^2 d\tilde{s}_d^2$ the rescaled horizon metric. It is the metric of the horizon embedded into the spacetime.

The first point we can make in the generalised case is that whilst the scalar curvature of B rescaled by $d\tilde{s}_d^2 \to r_+^2 d\tilde{s}_d^2$ is small, going as $\tilde{R} \to d(d-1)/r_+^2$, we will now construct another curvature invariant which is not vanishingly small for large r_+.

For compact manifolds with Einstein metrics, the Lichnerowicz spectrum on transverse tracefree modes is effectively bounded below by the Weyl tensor. Let λ be the minimum Lichnerowicz eigenvalue. Let $\kappa(x)$ be the, position dependent, eigenvalue of the Weyl tensor

$$
\tilde{C}_{\alpha\beta}{}^{\gamma\delta}(x)\tilde{h}_{\gamma\delta}(x) = \kappa(x)\tilde{h}_{\alpha\beta}(x)\,,
\tag{5.3}
$$

such that $\kappa_0 \equiv \| \kappa(x) \|_\infty = \sup_{x\in B} \kappa(x)$ is maximised. Think of this as the largest eigenvalue of the Weyl tensor. Then one has [16]

$$
\lambda \geq 4d - 4\kappa_0\,.
\tag{5.4}
$$

For the unstable horizon manifolds, we have seen that λ is large and negative and therefore κ_0 is large and positive and the $4d$ term is negligible. Further, we found above in (3.5) that $|\lambda| \gtrsim H^2 r_+^2$. Putting these results together allows us to bound a curvature invariant

$$
C_0 \equiv \| \tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta} \|_\infty \geq \kappa_0^2 \geq \frac{\lambda^2}{16} \gtrsim H^4 r_+^4\,.
\tag{5.5}
$$

Now, under the rescaling $d\tilde{s}_d^2 \to r_+^2 d\tilde{s}_d^2$, one has $\tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta} \to r_+^{-4}\tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta}$. But this now implies that $C_0 \to r_+^{-4}C_0 \sim H^4$, which is finite! This behaviour should be contrasted with the behaviour of the rescaled scalar curvature which becomes small, as noted above.

Two short comments are appropriate. Firstly, note that the rescaled $\tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta}$ is of order H^4 which, although finite, is small. It is easy to check that this implies that curvatures of the full black hole solution are small outside the horizon, if r_+ is large, and therefore we are within the regime of validity of the supergravity description. Secondly, we are effectively considering a double scaling limit. Both the horizon radius and the absolute value of the Lichnerowicz eigenvalue are large, but the relationship (5.4) essentially tells us that the limit is taken in such a way that $\tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta}$ remains finite at the horizon.

The persistence of a finite curvature invariant as the horizon manifold is rescaled suggests that a translationally invariant black brane picture is problematic. To see to what extent this is the case, consider the example of Böhm metrics, mentioned at the end of section 3.

The rescaled horizon Böhm metrics [9, 16] are of the form

$$r_+^2 d\tilde{s}_d^2 = r_+^2 \left[d\theta^2 + a(\theta)^2 d\Omega_p^2 + b(\theta)^2 d\widetilde{\Omega}_q^2 \right], \tag{5.6}$$

where $d\Omega_p^2$ and $d\widetilde{\Omega}_q^2$ are round metrics on S^p and $\tilde{S}^q$, respectively. Clearly $p+q+1=d$. The functions $a(\theta), b(\theta)$ are determined by the Einstein condition. Let us restrict attention to the cases where $p=q=2$ and the topology is $S^3 \times S^2$. Other cases are essentially the same, including when the topology is that of a sphere. In the $S^3 \times S^2$ case, $a(\theta)$ vanishes at $\theta = 0$ and $\theta = \theta_{\rm max}$, and $b(\theta)$ does not vanish. Both a and b are symmetric about the midpoint $\theta_{\rm max}/2$.

The finite curvature, when r_+ is large, is concentrated around $\theta = 0, \theta_{\rm max}$. Near $\theta = 0$ we have [16]

$$a(\theta) = \theta - \frac{2b_0^2+1}{18b_0^2}\theta^3 + \cdots ; \qquad b(\theta) = b_0 - \frac{4b_0^2-1}{6b_0}\theta^2 + \cdots . \tag{5.7}$$

There is a sequence of such metrics, given by a sequence of allowed values for b_0. The first is the standard metric on $S^3 \times S^2$. For the other metrics, b_0 becomes increasingly small. Thus, if we think of (5.6) as $\tilde{S}^2$ fibred over S^3 then as $b_0 \to 0$ the base S^3 starts to look like a higher dimensional rugby ball (or American football) and the fibre $\tilde{S}^2$ is small at the endpoints $\theta = 0, \theta_{\rm max}$. In the limit $b_0 = 0$ one obtains a singular "double-cone" over $S^2 \times S^2$. This may be quantified by calculating the rescaled curvature invariants

$$\begin{aligned} \tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta}\Big|_{\rm rescaled} &= \frac{8-16b_0^2+48b_0^4}{r_+^4 b_0^4} \quad \text{at} \quad \theta = 0\,, \\ \tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta}\Big|_{\rm rescaled} &= \frac{8-16b_0^2+8b_0^4}{r_+^4 b_0^4} \quad \text{at} \quad \theta = 0\,. \end{aligned} \tag{5.8}$$

These are in fact the maximum values obtained by the curvatures on the manifold. We have just seen that instability requires, for the rescaled curvatures, $\tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta} \sim H^4$, and therefore $b_0 \sim 1/(Hr_+)$.

In contrast to the behaviour at the endpoints, if we expand a and b about the midpoint $\theta_{\rm max}/2$ and calculate the rescaled curvatures we find

$$\tilde{R}_{\alpha\beta\gamma\delta}\tilde{R}^{\alpha\beta\gamma\delta}\Big|_{\rm rescaled} \sim \tilde{C}_{\alpha\beta\gamma\delta}\tilde{C}^{\alpha\beta\gamma\delta}\Big|_{\rm rescaled} \sim \frac{\mathcal{O}(1)}{r_+^4} \quad \text{at} \quad \theta = \frac{\theta_{\rm max}}{2}\,, \tag{5.9}$$

which is small for $r_+ \gg 1$. Because we are working with a riemannian metric, $\tilde{R}_{\alpha\beta\gamma\delta}$ $\tilde{R}^{\alpha\beta\gamma\delta} \to 0$ is equivalent to $\tilde{R}_{\alpha\beta\gamma\delta} \to 0$, which then implies that the space is becoming flat.

Therefore, if we are located away from the endpoints of the Böhm metrics, the large black hole limit does correspond locally to a black brane limit with a flat, translationally invariant, horizon. However, if we are located very near the endpoints, then a finite curvature persists in the large black hole limit, and the horizon may not be locally approximated by a metric with noncompact, translationally invariant directions.

Given these observations, it would be misleading to describe the large black hole limit as a black brane limit, although it is a locally accurate description away from the endpoints. Furthermore, the precise "ballooning" instability of the Böhm metrics [16], in which one of the S^2s grows and the other shrinks, crucially requires the finite curvature. This should not be surprising given that when there is no curvature, there is no instability. Thus, the strict statement of the Gubser-Mitra conjecture, requiring translational invariance, remains intact.

However, the results presented here suggest that one should be cautious about the general validity of using local thermodynamic arguments to understand classical instabilities.

6. Field theory implications of the instability

The generalised black holes we have been studying have a dual description using the logic of the AdS/CFT correspondence. A generalised AdS black hole with horizon manifold B will be dual to a thermal field theory on $S^1 \times B'$ [4, 10], where B' is a scaled up copy of B, to be specified shortly. We should be more precise about the size of the S^1 and B', that is, the dual temperature and spatial volume.

The temperature of the dual field theory at cutoff radius $r = r_0$ is given by the local, redshifted, temperature of the black hole. For $r_0 \gg r_+$ one has in the large black hole limit, using (5.1),

$$T_{\mathrm{FT}} = \frac{T}{\sqrt{-g_{tt}}} \approx \frac{T}{Hr_0} \approx \frac{(d+1)}{4\pi r_0} H r_+ \,. \tag{6.1}$$

The large black hole limit is thus the high temperature limit.

The volume of the spatial section at $r = r_0$ is just $\mathrm{Vol}(B') = r_0^d \mathrm{Vol}(B)$. Thus metrically we have $ds^2_{B'} = r_0^2 ds^2_B$.

The instability criterion of (3.5) may be rewritten in terms of the field theory temperature. The solution is unstable if

$$-\lambda' = -\frac{\lambda}{r_0^2} \gtrsim T_{\mathrm{FT}}^2 \frac{16\pi^2(0.86 + 2.6d + 0.33d^2)}{(d+1)^2} \,, \tag{6.2}$$

where λ' is the Lichnerowicz eigenvalue on B' corresponding to λ on B. Note that the final expression contains no explicit reference to the cutoff radius r_0. The criterion is expressed purely in terms of the Lichnerowicz spectrum of a manifold B' and the temperature of a field theory on that manifold.

Therefore, the duality predicts some effect in the thermal field theory on B' when the temperature drops below a certain value. The effect could be called a finite curvature effect because the previously discussed relationship between λ and the Weyl squared curvature invariant (5.5) implies that when (6.2) is satisfied, the temperature is small compared to the curvature of B' in certain regions.

The precise nature of the effect remains elusive for several reasons. Firstly, Böhm metrics currently provide the only known horizon manifolds that give unstable large AdS black holes, and only exist in dimensions $d = 5 \dots 9$. Therefore, the immediate application to a known duality is the $AdS_7 \times S^4$ version of the AdS/CFT correspondence. However, the field theory dual is not under computational control in this case. The question of finding unstable large black holes in lower dimensional generalised AdS spaces, AdS_5 and AdS_6, depends on the existence of suitable positive curvature Einstein manifolds in three and four dimensions, respectively. In three dimensions, positive curvature Einstein manifolds are quotients of the round sphere [25] and therefore cannot have negative Lichnerowicz eigenvalues. In four dimensions, the situation is not clear, although some rigidity results exist [25, 26].

Secondly, given that the unstable generalised black holes cannot provide the dual description to field theory physics below the critical temperature, new stable gravity solutions are needed. It is not known whether such solutions, stable and with appropriate $S^1 \times B'$ asymptotics, exist or what properties they might have. Understanding whether these stable solutions exist could also shed some light upon the endpoint of the classical instability.

There are also various issues concerning the thermal field theory which one would like to understand better. Is the effect specific to exotic field theories such as that dual to M-theory on $AdS_7 \times S^4$ or is it generic? Does the spatial background B' need to be Einstein, or is it sufficient to have a Lichnerowicz spectrum satisfying (6.2)?

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JHEP08(2003)019